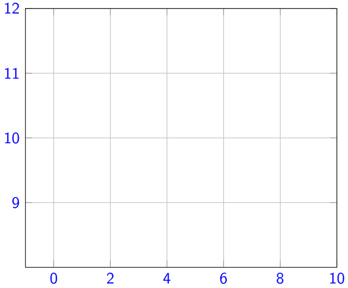
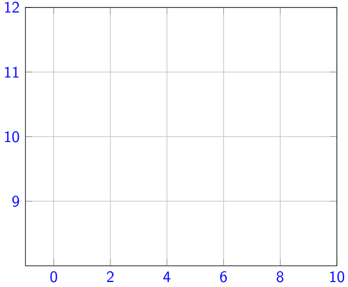
Chapter 4.1: Extreme Values of Functions

Maximum and Minimum

- f has an *absolute maximum* at c if $f(x) \leq f(c)$ for all x.
- f has an *absolute minimum* at c if $f(x) \geq f(c)$ for all x.
- f has an *local maximum* at c if $f(x) \leq f(c)$ for all x *near* c.
- f has an *local minimum* at c if $f(x) \geq f(c)$ for all x *near* c.





Don't forget flat line and open interval.

Derivatives Help

- If $f'(c) > 0$, then near c our function f is going up.
- If $f'(c) < 0$, then near c our function f is going down.
- *If $f'(c) \neq 0$ and c is NOT ON THE BOUNDARY, then c is not a local min or max.*

A point c is *critical point* if

- $f'(c)$ is undefined
- $f'(c) = 0$
- c is a boundary point

A (local) extreme can only occur at a critical point.

Example: Find all critical points of $x^{2/3}e^{-x/3}$ for $-1 \leq x \leq 5$.

Make derivative = 0.

$$\frac{2}{3}x^{-1/3}e^{-x/3} + x^{2/3}e^{-x/3}(-1/3) = 0$$

$$2x^{-1/3} - x^{2/3} = 0$$

$$\frac{1}{\sqrt[3]{x}}(2 - x) = 0$$

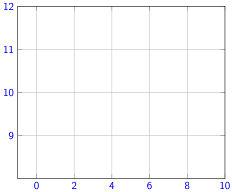
Critical points are $\{-1, 5, 0, 2\}$.

Existence of Extreme Points

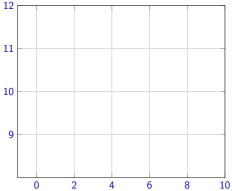
Extreme Value Theorem

A continuous function on a closed and bounded interval (e.g. $a \leq x \leq b$) has an absolute min and absolute max.

Continuous is necessary



Closed and bounded in necessary



If we have a continuous function f on a closed bounded interval, we can find the absolute max and absolute min by the following procedure:

- Find all critical points
- Evaluate f at critical points
- Larges value = absolute max
  smallest value = absolute min

Example: Find absolute min and max of $f(x) = x^2 - x$ on $[0, 1]$

The derivative is $f'(x) = 2x - 1$. The only critical point is $c = 1/2$. Now, we evaluate the function at 0, 1/2, and 1:

$$f(0) = 0, \quad f(1/2) = -1/4 \quad f(1) = 0$$

Consequently, the absolute maximum **value** is $0 = f(0) = f(1)$ and the absolute minimum **value** is $-1/4 = f(1/2)$

Examples

Find absolute max and min of

$g(x) = 2x^3 - 9x^2 + 12x + 6$ on $[2, 3]$

Take the derivative:

$$g'(x) = 6x^2 - 18x + 12 = 6(x^2 - 3x + 2)$$
$$= 6(x-1)(x-2)$$

The only critical point *in the interval* $[2, 3]$ is $x = 2$. Thus we need only check the endpoints:

$$g(2) = 10 \quad \text{and} \quad g(3) = 15$$

So the absolute maximum value is $15 = g(3)$ and the absolute minimum value is $10 = g(2)$.

$f(x) = \sqrt{4 - x^2}$ on $[-2, 1]$

The derivative is

$$f'(x) = -\frac{x}{\sqrt{4 - x^2}}$$

The critical points in the domain are $c = 0, -2$. So, we evaluate the function at -2, 0, and 1:

$$f(-2) = 0 \quad f(0) = 2 \quad f(1) = \sqrt{3}$$

Thus the absolute maximum value is $2 = f(0)$ and the absolute minimum is $0 = f(-2)$.

Examples

Find absolute max and min of

$f(x) = |x^2 - 4x - 5|$ for $0 \leq x \leq 6$

No derivative at $x^2 - 4x - 5 = 0$. That is $(x-5)(x+1) = 0$. So critical points are 5 and -1.

Then for finding derivatives $= 0$, we can try $x^2 - 4x - 5$ and take the derivative there: $2x - 4 = 0$. Hence critical point is $x = 2$.

All critical points are $\{0, 2, 5, 6\}$. We evaluate f at critical points and get

$$f(0) = 5; f(2) = 9; f(5) = 0; f(6) = 7$$

Hence absolute max is $f(2) = 9$ and absolute min is $f(5) = 0$.

$g(t) = t^8 e^{-t^2}$ for $-1 \leq t \leq 10$

Take the derivative $= 0$

$$8t^7 e^{-t^2} - t^8 e^{-t^2} 2t = 0$$

If $t \neq 0$ we get

$$8 - 2t^2 = 0$$

Hence $t \in \{-2, 2\}$.Note -2 is not in $[-1, 10]$. Now critical points are $\{-1, 0, 2, 10\}$. Lets evaluate $g(t)$ at critical points.

$f(-1) = e^{-1}$; $f(0) = 1$; $f(2) = 2^8 \cdot e^{-4}$; $f(10) = 2^{10} \cdot e^{-100}$

Absolute max is $f(2) = 2^8 \cdot e^{-4}$.

Absolute min is $f(2) = f(10) = 2^{10} \cdot e^{-100}$.

Examples

Find absolute max and min of $g(x) = 1/(x - 3/4) + \ln(x)$ on $[1, 4]$.

Take the derivative:

$$g'(x) = -\frac{1}{(x - 3/4)^2} + \frac{1}{x}$$

These derivatives do not exist at 0 and 3/4, but we do not need to worry about these points. Set it equal to zero, then

$$\begin{aligned}
0 &= -\frac{1}{(x-3/4)^2} + \frac{1}{x} \\
\frac{1}{(x-3/4)^2} &= \frac{1}{x} \\
x &= (x - 3/4)^2 \\
x &= x^2 - \frac{3}{2}x + \frac{9}{16} \\
0 &= x^2 - \frac{5}{2}x + \frac{9}{16} \\
0 &= \frac{1}{16}(4x-1)(4x-9)
\end{aligned}$$

The critical points are 1/4 and 9/4, and we need only to investigate 9/4. So, we evaluate the function at 1, 9/4, and 4:

$g(1) = 4 \quad g(9/4) = 2/3 + \ln(9/4) \approx 1.4$
$g(4) = 4/13 + ln(4) \approx 1.6$

Thus the absolute maximum value is $4 = g(1)$ and the absolute minimum value is $2/3 + \ln(9/4) = g(9/4)$.